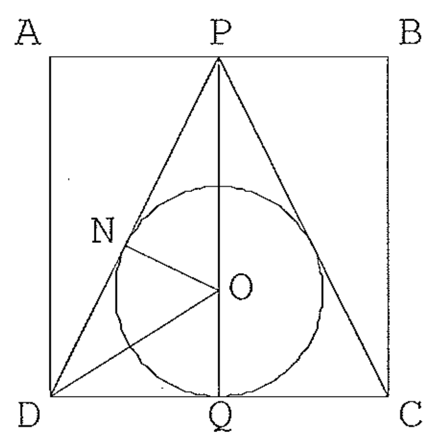
Q1. An isosceles triangle is inscribed inside a square $ABCD$ of side length l such that the non-equal side of the triangle is one side of the square. A circle is inscribed inside the triangle. What is the area of the circle?

Solution. Using the labels in the illustration below, let $r = OQ = ON$ be the radius of the circle. By SSS $\Delta APD \cong \Delta BPC$ so that $AP = PB = \frac{l}{2}$. Let Q be the midpoint of DC, so PQ is perpendicular to DC and $DQ = \frac{l}{2}$. From the Pythagorean theorem we see that the length of PD is $\sqrt{l^2 + \left(\frac{l}{2}\right)^2} = \frac{l\sqrt{5}}{2}$. Since $\Delta OND \cong \Delta OQD$ by SSS, the length of ND is $\frac{l}{2}$, which implies that the length of PN is $\frac{(\sqrt{5}-1)l}{2}$. Again using the Pythagorean theorem on the right triangle ΔPNO gives $\left(\frac{(\sqrt{5}-1)l}{2}\right)^2 + r^2 = (l-r)^2$, that is, $\left(\frac{(\sqrt{5}-1)}{2}\right)^2 l^2 = l^2 - 2rl$. Solving for r gives

$$r = \frac{1}{2l}\left[l^2 - \left(\frac{\sqrt{5}-1}{2}\right)^2 l^2\right] = \frac{l}{8}\left[4 - \left(\sqrt{5}-1\right)^2\right] = \frac{l\left(\sqrt{5}-1\right)}{4}.$$ Therefore the area of the circle is $$\pi\left[\frac{l\left(\sqrt{5}-1\right)}{4}\right]^2 = \frac{l^2\pi\left(3-\sqrt{5}\right)}{8}.$$



Q2. "I actually had a seat in the bus," said Joan. "There weren't that many people and we started with only 18 passengers."

Andy smiled, "I guess it filled up on the way."

"Only a bit," his wife replied. "The first stop, half got out and some got in. At the next stop, again half got out, but one less got in than at the first stop. It is funny that at each stop after that, exactly half got out, and always one fewer got in than at the previous stop. When we reached my stop there were 39 passengers in the bus."

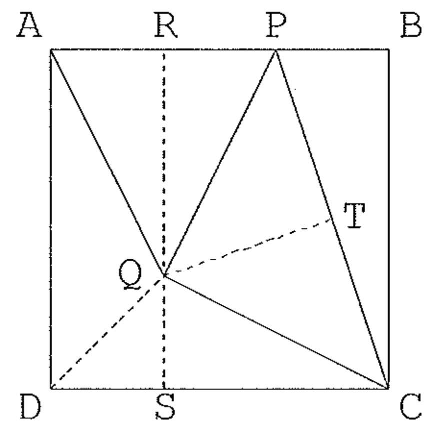
How many got in at the first stop?

Solution. Start with 18 passengers and let k be the number of passengers that got in at the first stop. Then after stop one there are $18-\frac{18}{2}+k=9+k$ passengers on the bus. After stop two there are $9+k-\frac{9+k}{2}+k-1=\frac{7+3k}{2}$ passengers on the bus. Notice if $\frac{7+3k}{2}=39$ then k would not be a natural number, a contradiction. Continue in this fashion, after stop three there are $\frac{7+3k}{4}+k-2=\frac{7k-1}{4}$ passengers on the bus and this cannot equal 39. After stop four there are $\frac{7k-1}{8}+k-3=\frac{15k-25}{8}$ and this cannot equal 39. After stop five there are $\frac{15k-25}{16}+k-4=\frac{31k-89}{16}=39$ when $k=\frac{39\cdot 16+89}{31}=23$. Thus the answer is 23 passengers got in at the first stop.

Q3. The square $ABCD$ has side length three. The point P splits side AB such that $|AP|:|PB|=2:1$. A point Q inside the square is chosen such that $|AQ|=|PQ|=|CQ|$. Find the area of ΔCPQ.

Solution. Let R be the midpoint of AP. Since $\Delta ARQ\cong\Delta PRQ$ by SSS, we see that $\angle ARQ$ is a right angle. Similarly, if T is the midpoint of PC then $\angle PTQ$ is also a right angle. By the theorem of Pythagoras, since $PB=1$ and $BC=3$ we see that $PC=\sqrt{10}$. Also by SSS we have $\Delta AQD\cong\Delta CQD$ so that $\angle ADQ=45^\circ$. Continue the line RQ until it intersects DC at S, then $\angle QSD$ is a right angle with $DS=1$. Thus $DQ=\sqrt{2}$, $SQ=1$ and $RQ=2$. Using the Pythagorean theorem again yields $PQ=\sqrt{5}$ and

$QT = \sqrt{5/2}$. Thus the area of ΔCQP is $\frac{1}{2} \cdot \frac{\sqrt{5}}{\sqrt{2}} \cdot \sqrt{10} = \frac{5}{2}$.



Q4. The product of three positive integers is 825. Determine all possible sums of the three integers if the second integer is eleven times the first integer.

Solution. Let $xyz = 825$. By assumption, $y = 11x$ and also $825 = 11 \cdot 5^2 \cdot 3$, so the equation becomes $11x^2 z = 11 \cdot 5^2 \cdot 3$. Thus $x^2 z = 5^2 \cdot 3$, which implies that $x = 1$ or $x = 5$. When $x = 1$ we get $y = 11$ and $z = 75$, so that $x + y + z = 87$. When $x = 5$ we have $y = 55$ and $z = 3$, which gives $x + y + z = 63$.

Q5. Find all natural numbers b such that $\sqrt{30+\sqrt{b}} + \sqrt{30-\sqrt{b}}$ is a natural number.

Solution. Let b be a natural number and set $\sqrt{30+\sqrt{b}} + \sqrt{30-\sqrt{b}} = n$ for a natural number n. Then by squaring both sides we see that $60 + 2\sqrt{30^2 - b} = n^2$, which implies that $2\sqrt{30^2 - b} = n^2 - 60$. Note that this implies that $n^2 \geq 60$ and that n^2, hence n, is even. Also we see that $b = 30^2 - \left(\frac{n^2 - 60}{2}\right)^2$ which implies that $30^2 - \left(\frac{n^2 - 60}{2}\right)^2 > 0$ so that $n^2(120 - n^2) > 0$. Thus we have $60 \leq n^2 < 120$, which gives $n = 8$ or $n = 10$ as the only possibilities for n. Thus the only possibilities for b are 500 or 896.

Q6. Let $E(x) = \frac{9^x}{9^x + 3}$. Find $E(x) + E(1 - x)$ and use this to evaluate

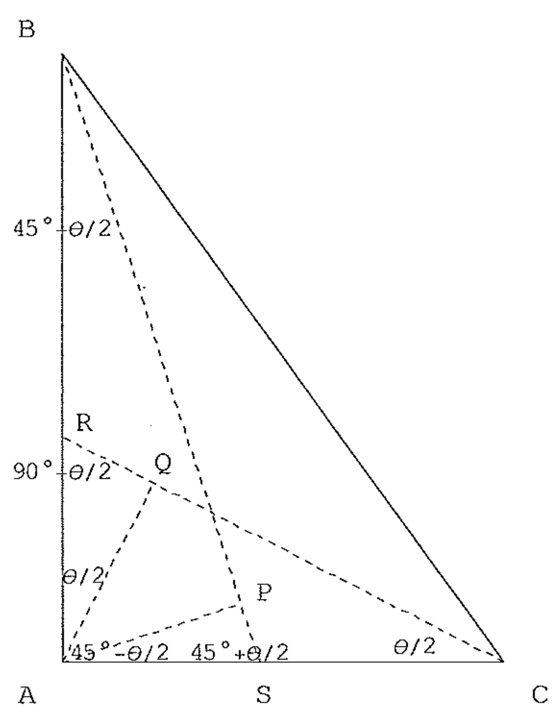
$$E\left(\frac{1}{2012}\right)+E\left(\frac{2}{2012}\right)+\cdots+E\left(\frac{2010}{2012}\right)+E\left(\frac{2011}{2012}\right).$$

Solution. We notice that $E(1-x)=\frac{9^{1-x}}{9^{1-x}+3}=\frac{9}{9+3\cdot 9^x}=\frac{3}{3+9^x}$, so that $E(x)+E(1-x)=1$. This implies that

$$E\left(\frac{1}{2012}\right)+E\left(\frac{2}{2012}\right)+\cdots+E\left(\frac{2010}{2012}\right)+E\left(\frac{2011}{2012}\right)$$
$$=E\left(\frac{1}{2012}\right)+E\left(\frac{2011}{2012}\right)+E\left(\frac{2}{2012}\right)+E\left(\frac{2010}{2012}\right)+\cdots+E\left(\frac{1005}{2012}\right)+E\left(\frac{1007}{2012}\right)+E\left(\frac{1006}{2012}\right)$$
$$=1+1+\cdots+1+E\left(\frac{1006}{2012}\right)=1005+E\left(\frac{1}{2}\right)=1005+\frac{\sqrt{9}}{\sqrt{9}+3}=1005+1/2.$$

Q7. Triangle ΔABC has $\angle BAC=90^\circ$. The feet of the perpendiculars from A to the internal bisectors of $\angle ABC$ and $\angle ACB$ are P and Q, respectively. Find the measure of $\angle PAQ$.

Solution. Using the labels as in the below illustration, let $\theta=\angle ACB$, so that $\angle ABC=90^\circ-\theta$. Thus $\angle ACR=\theta/2$ and $\angle ABS=45^\circ-\theta/2$. Therefore $\angle ARC=90^\circ-\theta/2$ and $\angle ASB=45^\circ+\theta/2$. This gives $\angle RAQ=\theta/2$ and $\angle PAC=45^\circ-\theta/2$, which provides the answer being $\angle QAP=45^\circ$.



Q8. Solve the equation for x: $8(4^x+4^{-x})-54(2^x+2^{-x})+101=0$.

Solution. Let $y=2^x$, then the equation becomes $8(y^2+y^{-2})-54(y+y^{-1})+101=0$ or $8(y+y^{-1})^2-54(y+y^{-1})+101-16=0$. Letting $q=y+y^{-1}$, the equation becomes $8q^2-54q+85=0$, which factors nicely as $(2q-5)(4q-17)=0$. Thus $q=17/4=y+y^{-1}$ or $q=5/2=y+y^{-1}$. Solving each equation for y gives $y=\frac{1}{4}=2^x$, $y=4=2^x$, $y=\frac{1}{2}=2^x$ or $y=2=2^x$. Solving for x gives the final answer, $x=-2$, $x=2$, $x=-1$ or $x=1$.

Q9. Let $m>0$, find all real solutions to the equation $m+\sqrt{m+\sqrt{m+\sqrt{m+\sqrt{x}}}}=x$.

Solution. We first solve the equation $m+\sqrt{x}=x$, that is, $0=x-\sqrt{x}-m$. Let $u=\sqrt{x}\geq 0$, the solution to $0=u^2-u-m$ is $u=\frac{1+\sqrt{1+4m}}{2}=\sqrt{x}$, where we chose the plus square to ensure u is non-negative. Therefore the only solution to $m+\sqrt{x}=x$ is $x=\left(\frac{1+\sqrt{1+4m}}{2}\right)^2=\frac{2+4m+2\sqrt{1+4m}}{4}=\frac{1+2m+\sqrt{1+4m}}{2}$. Next notice that $m+\sqrt{x}=x$ is an equivalent equation to the equation $m+\sqrt{m+\sqrt{x}}=m+\sqrt{x}$, hence has the same solution. To check, let $u=\sqrt{m+\sqrt{x}}\geq 0$, the new equation becomes, once again, $0=u^2-u-m$. Thus we get $u=\frac{1+\sqrt{1+4m}}{2}=\sqrt{m+\sqrt{x}}$ or

$$\sqrt{x}=-m+\left(\frac{1+\sqrt{1+4m}}{2}\right)^2=-m+\frac{1+2m+\sqrt{1+4m}}{2}=\frac{1+\sqrt{1+4m}}{2},$$

so once again we have $x=\frac{1+2m+\sqrt{1+4m}}{2}$. So no matter how many times we repeat the process, the solution will always be $x=\frac{1+2m+\sqrt{1+4m}}{2}$.

Q10. A bubble tower with three chambers consists of a spherical bubble of radius one which is surmounted by a smaller, hemispherical bubble, which in turn is surmounted by another smaller, hemispherical bubble, as in the below illustration. What is the maximum height of any bubble tower with three chambers?

Solution. We use (twice) the identity $A\sin(\theta)+B\cos(\theta)=\sqrt{A^2+B^2}\sin(\theta+\psi)$, where $\tan(\psi)=B/A$. Using the labels as in the illustration below, the height of a bubble tower with three chambers becomes $1+h_1+h_2+h_3$. Now $h_1=r_1=r_2\cos(\theta_1)$, $h_2=r_2\sin(\theta_1)$,

$h_3 = \sin(\theta_2)$ and $r_2 = \cos(\theta_2)$. Thus
$h_1 + h_2 = r_2[\cos(\theta_1) + \sin(\theta_1)] = r_2\sqrt{2}\sin(\theta_1 + \pi/4) = \sqrt{2}\cos(\theta_2)\sin(\theta_1 + \pi/4)$, which gives
$h_1 + h_2 + h_3 = 1\sin(\theta_2) + \sqrt{2}\sin(\theta_1 + \pi/4)\cos(\theta_2) = \sqrt{1 + 2\sin^2(\theta_1 + \pi/4)}\sin(\theta_2 + \psi)$, where
$\tan(\psi) = \sqrt{2}\sin(\theta_1 + \pi/4)$. Therefore $h_1 + h_2 + h_3 \le \sqrt{1 + 2\sin^2(\theta_1 + \pi/4)} \le \sqrt{3}$. For fun, note to get equality in the second inequality one needs $\theta_1 = \pi/4$, which gives $\tan(\psi) = \sqrt{2}$. To get equality in the first inequality one needs $\theta_2 = \pi/2 - \psi$.