1. Find all solutions to $\sqrt{1+x} = x + \sqrt{1-x}$ and provide all algebra for full credit.

   Solution: Squaring both sides of the given equation gives $1 + x = x^2 + 2x\sqrt{1-x} + 1 - x$ which implies $2x - x^2 = 2x\sqrt{1-x}$. This gives the possibility of x as a solution. Checking we have $\sqrt{1+0} = 1$ and $0 + \sqrt{1-0} = 1$ and we then have $x = 0$ as a solution. If we assume $x \neq 0$ and divide both sides by x, we then have $2 - x = 2\sqrt{1-x}$ which then gives $4 - 4x + x^2 = 4 - 4x$ and hence $x^2 = 0$. The only solution is $x = 0$.

   □

2. Let $x = \underbrace{111\ldots11}_{2017\ 1's}$ and $y = 1\underbrace{000\ldots00}_{2016\ 0's}5$. Determine, with justification, $\sqrt{xy+1}$.

   Hint: think "geometrically."

   Solution: Write $x = \sum_{k=0}^{2016} 10^k = \frac{10^{2016}-1}{10-1} = \frac{10^{2016}-1}{9}$ and $y = 10^{2016} + 5$. Then

$$\begin{aligned}\sqrt{xy+1} &= \sqrt{\frac{(10^{2016}-1)(10^{2016}+5)+9}{9}} \\ &= \sqrt{\frac{10^{2\cdot 2016} + 4\cdot 10^{2016} + 4}{9}} \\ &= \sqrt{\left(\frac{10^{2016}+2}{3}\right)^2} \\ &= \frac{10^{2016}+2}{3}.\end{aligned}$$

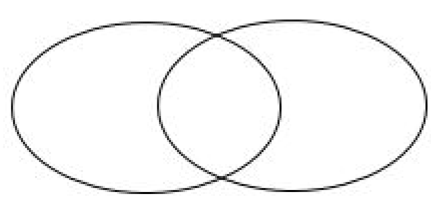
   □

3. Find the number x and the digit symbolized by d in $(3(230+x))^2 = 492, d04$.

   Solution: Since 9 divides the left-hand side, 9 must divide the right-hand side and so the sum of the digits on the right-hand side must be divisible by 9. Therefore $4 + 9 + 2 + d + 0 + 4 = d + 19$ must be divisible by 9. The only digit that makes this true is $d = 8$. We then solve for x in $(3(230+x))^2 = 492804$. This implies $9(230+x)^2 = 492804$ and so $(230+x)^2 = 54756$. Hence $230 + x = 234$ or $230 + x = -234$ giving the possibilities of $x = 4$ or $x = -464$.

   □

4. Mr. Chane was walking in Linkin Park when he saw a chain with 2 links which measured 12 cm in length. Continuing his journey, he then spied a chain with 5 links which was 25 cm long. Continuing, Mr. Chane then noticed a chain with 2017 links. All links were the same in all of the chains with the exact same "overlap" of two consecutive links. How long was the chain with 2017 links?



Solution: Assume d is the length of one link in the chain and that v is the overlap of two links. Then we have

$$2d - v = 12$$
$$5d - 4v = 25.$$

Solving this system of equations gives $d = \dfrac{23}{3}$ cm and $v = \dfrac{10}{3}$ cm. In a chain with 2017 links, the length would then be

$$2017d - 2016v = 2017\left(\frac{23}{3}\right) - 2016\left(\frac{10}{3}\right) = \frac{26231}{3} = \left(8743 + \frac{2}{3}\right) \text{ cm.}$$

□

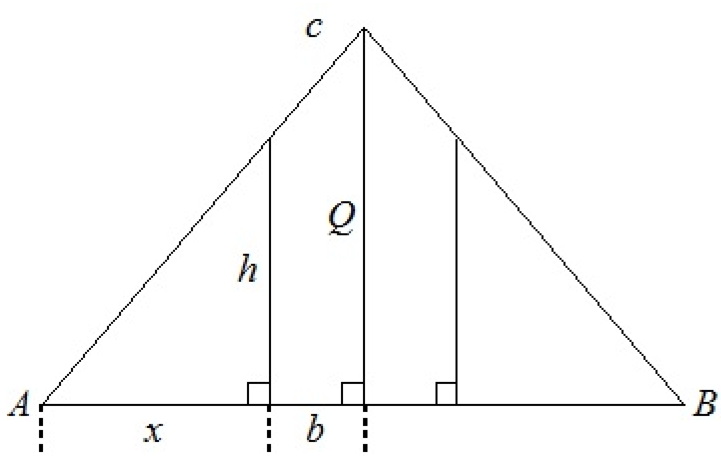
5. Find the number of solutions (x, y) in non-negative integers to the equation $x^2y^3 = 6^{12}$. Note: if you can solve this problem, you will be "primed" for success!

   Solution: Since $6 = 2 \cdot 3$, we can rewrite the given equation as $x^2y^3 = 2^{12}3^{12}$. Hence $x = 2^a3^b$ and $y = 2^k3^l$ for some non-negative integers a, b, k, l which implies $x^2 = 2^{2a}3^{2b}$ and $y^3 = 2^{3k}3^{3l}$. Therefore, $2a + 3k = 12$ and $2b + 3l = 12$ and we conclude $(a, k) \in \{(0, 4), (3, 2), (6, 0)\}$ and $(b, l) \in \{(0, 4), (3, 2), (6, 0)\}$. Any of the three values of a can be paired with any of the three values of b which then determine the values of k and l, so there are $3 \cdot 3 = 9$ solutions in non-negative integers.

   □

6. In the following figure, an isosceles triangle with $AB = 12$ is divided into 4 polygons of equal area using segments perpendicular to AB. Find x.

   Solution: Label the picture as below.



To find x, we will compute the area of the quadrilateral $Qbhc$ in two different ways, set them equal and then solve the resulting equation. First, we are told that each polygon has equal area and the leftmost right triangle has area $\frac{1}{2}xh$, so the quadrilateral $Qbhc$ does as well. We can also compute the area of this quadrilateral by computing the area of the triangle, call it T, with height Q and base $x + b$ and subtracting xh. The area of T is $\frac{1}{2}Q(x + b)$. Since $x + b$ is half of the length of AB, we have $x + b = 6$. Therefore, the area of T is $3Q$.

Now, by similar triangles, $\dfrac{h}{x} = \dfrac{Q}{x+b}$ and so $Q = \dfrac{(x+b)h}{x} = \dfrac{6h}{x}$. Since $3Q = xh$, we have $3\left(\dfrac{6h}{x}\right) = xh$. Solving for x gives $x = 3\sqrt{2}$.

□

7. Solve the equation $2017^{\sqrt{x-2016}} = \dfrac{1}{x-2015}$. For full credit, justify why you have found all possible solutions.

   Solution: In order for $\sqrt{x-2016}$ to be defined, we must have $x \geq 2016$. Hence $x-2015 \geq 1$ and so $\dfrac{1}{x-2015} \leq 1$. Therefore, $2017^{\sqrt{x-2016}} \leq 1$. Since $f(t) = 2017^t$ is an increasing function, we must have $\sqrt{x-2016} \leq 0$. Therefore, $\sqrt{x-2016} = 0$ and so $x = 2016$.

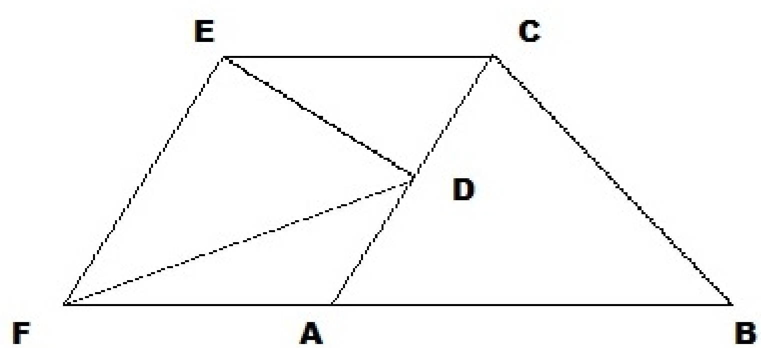
□

8. Evaluate $\cos^2(1°) + \cos^2(2°) + \ldots \cos^2(90°)$ without a calculator.

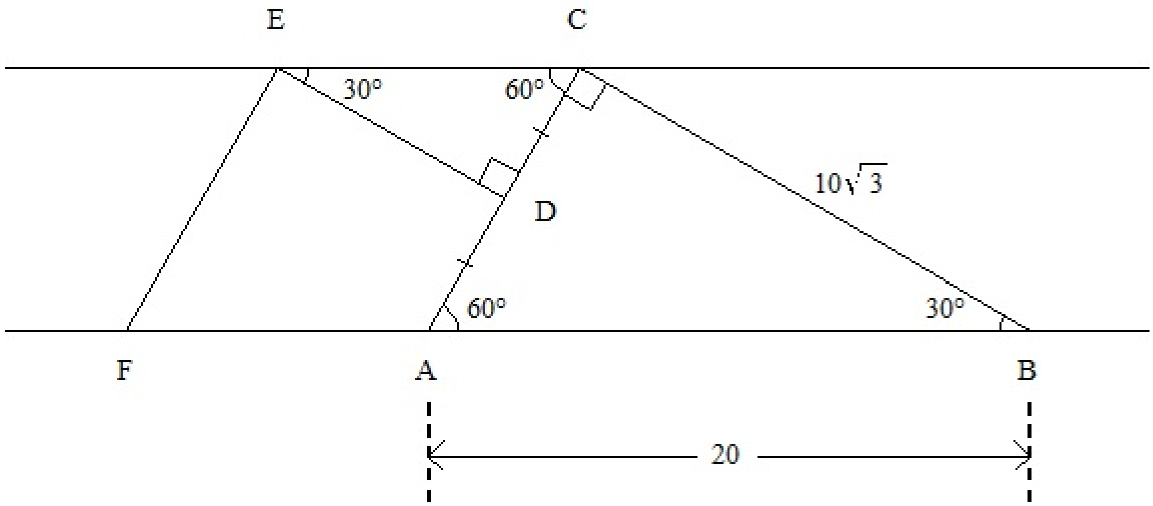
   Solution: First, notice that $\cos(90°) = 0$. Let $S = \sum_{k=1}^{89} \cos^2(k)$. Since $\cos(k) = \sin(90° - k)$, we have $S = \sum_{k=1}^{89} \sin^2(90° - k)$. Also since $\sin^2(x) = 1 - \cos^2(x)$, we have $S = \sum_{k=1}^{89}(1 - \cos^2(90° - k))$. Now, letting $q = 90° - k$, and substituting gives $S = \sum_{q=1}^{89}(1 - \cos^2(q)) = 89 - S$. Hence $2S = 89$ and $S = 44.5$. Note, one could also use the power reduction formula combined with the symmetry of the cosine function to solve this problem.

□

9. In the diagram below, triangle ABC is a $30° - 60° - 90°$ triangle with the right angle at vertex C, the $30°$ angle at vertex B, and side AB having length 20. Segment ED is perpendicular to side AC and D bisects AC. Segment EC is parallel to AB. Segment EF is perpendicular to ED and F is on the extension of AB. Find the length of segment ED. Note, the diagram below is not drawn very accurately.



   Solution: Here is a more accurate diagram.

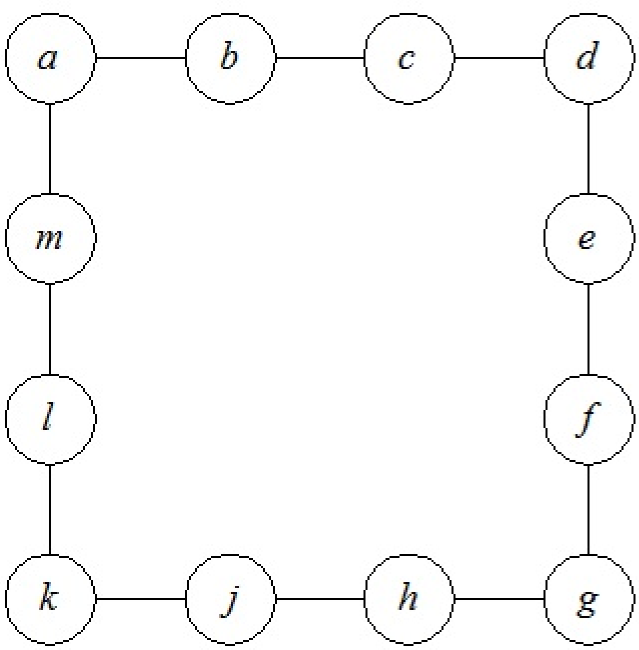


Triangle ABC is a $30° - 60° - 90°$ right triangle with hypotenuse 20, hence $AC = 10$. Since D bisects AC, we have $CD = 5$. Now, EC is parallel to AB and ED is perpendicular to AC, so we deduce that CDE is also a $30° - 60° - 90°$ right triangle with hypotenuse EC. Therefore $EC = 10$ and $ED = 5\sqrt{3}$.

□

10. Each of the integers from 1 to 12 is to be placed in one of the circles in the diagram below so that the sum of the integers along each side is 25. Determine the sum of the four integers placed in the corners.

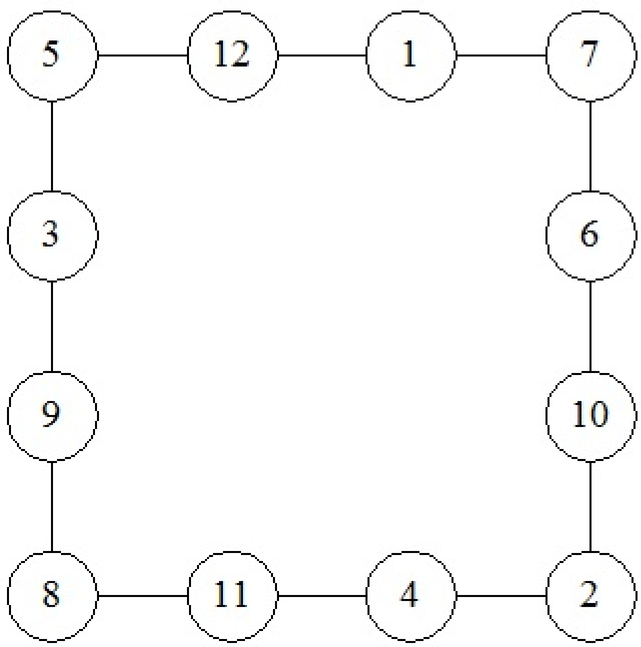
    Solution: Let $a, b, c, d, e, f, g, h, j, k, l, m$ be the twelve integers from 1 to 12 placed in each circle as shown below.



Then we must have the following system of equations.

$$\begin{aligned} a+b+c+d &= 25 \\ d+e+f+g &= 25 \\ g+h+j+k &= 25 \\ k+l+m+a &= 25 \end{aligned}$$

Adding these equations together gives $2a+b+c+2d+e+f+2g+h+j+2k+l+m = 100$. Hence $a+d+g+k+(1+2+\cdots+12) = 100$ giving us $a+d+g+k+78 = 100$ and so $a+d+g+k = 22$. This is the unique sum of the four integers in the corners. One possible filling is given below.



$\square$